Semester One Examination, 2023

Question/Answer booklet

MATHEMATICS APPLICATIONS UNIT 3

SOLUTIONS

Section Two: Calculator-assumed

WA student number: In figures | | | | | | | | |

In words _____

Your name _____

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Number of additional answer booklets used (if applicable): | |

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	12	12	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

See next page

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

See next page

Question 8 **(7 marks)**

A deputy principal collected data over a 24-hour period on hours of sleep s and hours of mobile phone use h from a random sample of students in their school. The coefficient of determination between the variables was calculated to be 0.85 and the equation of the least-squares line that fitted the data was $\hat{s} = -0.22h + 9.25$.

(a) State what percentage of the variation in the hours of sleep of these students can be explained by the variation in their hours of mobile phone use, and hence assess the strength of the association between these variables. (2 marks)

Solution
85% of the variation in s can be explained by the variation in h. This is a large percentage of the variation and so the association between the variables is strong.
Specific behaviours
✓ correct percentage ✓ assesses strength as strong

(b) Determine the correlation coefficient between s and h for this data. (2 marks)

Solution
$r_{sh} = -\sqrt{0.85} = -0.922$
Specific behaviours
✓ indicates square root of r^2 ✓ uses slope of line to obtain correct coefficient

(c) Use the least-squares line to predict the hours of sleep for a student who used their mobile phone for 7.5 hours over a 24-hour period. (1 mark)

Solution
$\hat{s} = -0.22(7.5) + 9.25 = 7.6$ h
Specific behaviours
✓ correct prediction

(d) Discuss the validity of the prediction in part (b). (2 marks)

Solution
The strong association between the variables supports its validity. However, the lack of raw data means it is not possible to determine if the prediction involves extrapolation and so the prediction should be treated with caution.
Specific behaviours
✓ indicates strong association supports validity ✓ indicates difficulty posed by lack of raw data

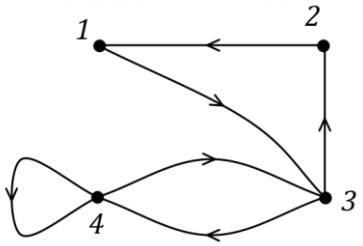
See next page

Question 9 **(7 marks)**

A harbour has four ferry stations, numbered $1, 2, 3$ and 4. A ferry can only be caught from 1 to 3, from 2 to 1, from 3 to 2, from 3 to 4 and from 4 to 3. There is also a harbour sightseeing ferry that starts and finishes at station 4.

(a) Draw directed graph G to represent the above information. (2 marks)

Solution



Specific behaviours

✓ four labelled vertices
✓ correct directed graph

(b) Construct $\boldsymbol{M}$, the adjacency matrix for G. (2 marks)

Solution

$$\boldsymbol{M} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Specific behaviours

✓ any 4×4 matrix filled with zeros and ones
✓ correct matrix

(c) Lolo bought an all-day ferry ticket, caught some ferries and ended up at station 1. Her journey formed a walk of length 5 in graph G. Form an appropriate multistage matrix and use it to explain which station(s) she could **not** have started from. (3 marks)

Solution

$$\boldsymbol{M}^5 = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 2 & 2 \\ 2 & 2 & 5 & 6 \\ 2 & 4 & 6 & 9 \end{bmatrix}$$

The zero in the first column indicates that there is no 5-stage journey starting from station 2 that ends at station 1, so she could not have started from station 2.

Specific behaviours

✓ calculates $\boldsymbol{M}^5$
✓ explanation using zero in first column
✓ states correct station

See next page

Question 10 **(8 marks)**

The weight of fish caught by anglers on the first and second day of a 7-day angling competition was 17.5 and 19.6 kg respectively.

Let the weight of fish caught on day n of the competition be T_n kg.

(a) Assume that the daily weights form an arithmetic sequence.

(i) What weight of fish was caught on the third day of the competition? (2 marks)

Solution
$d = 19.6 - 17.5 = 2.1$ kg $T_3 = 19.6 + 2.1 = 21.7$ kg
Specific behaviours
✓ indicates correct difference of terms ✓ correct weight

(ii) Deduce a rule that models the weight of fish caught on the n^{th} day of the competition. (1 mark)

Solution
$T_n = 17.5 + (n-1)(2.1)$ $= 2.1n + 15.4$
Specific behaviours
✓ correct n^{th} term rule (simplification not required)

(iii) What weight of fish was caught on the last day of the competition? (1 mark)

Solution
$T_7 = 17.5 + 6 \times 2.1 = 30.1$ kg
Specific behaviours
✓ correct weight

(b) Assume that the daily weights form a geometric sequence.

(i) Determine a recursive rule to model the daily weight of fish caught. (2 marks)

Solution
$r = 19.6 \div 17.5 = 1.12$ $T_{n+1} = 1.12T_n, \quad T_1 = 17.5$
Specific behaviours
✓ indicates correct ratio of terms ✓ recursive rule with term of sequence

(ii) To the nearest 0.1 kilogram, what weight of fish was caught on the last day of the competition? (2 marks)

Solution
$T_7 = 17.5(1.12)^{7-1} = 34.5$ kg
Specific behaviours
✓ indicates correct method ✓ correct weight

See next page

Question 11 **(8 marks)**

The balance a_n of savings account A after n monthly payments of $\$50$ have been made into it can be modelled using the following recurrence relation:

$$a_{n+1} = 1.0084a_n + 50, \qquad a_0 = 300.$$

Savings account A has an interest rate of 0.84% per month.

(a) State the balance of the savings account A before any monthly payments were made. (1 mark)

Solution
$\$300$
Specific behaviours
✓ correct balance

(b) Determine the balance of the savings account A after 15 monthly payments have been made. (2 marks)

Solution
$A_{15} = \$1135.85$
Specific behaviours
✓ correct balance to nearest dollar ✓ balance in dollars, rounded to 2 dp

(c) Determine k, the number of monthly payments that are required for the balance of savings account A to first exceed $\$4000$, and state the value of a_k. (2 marks)

Solution
$k = 56, \qquad a_{56} = 4035.75$
Specific behaviours
✓ correct value of k ✓ corresponding value of a_k

The recurrence relation $b_{n+1} = 1.0052b_n + 44,\ b_0 = 500$, models the balance b_n of a similar savings account B after n monthly payments of $\$x$ have been made into it.

(d) State the value of x, the monthly payment made into savings account B. (1 mark)

Solution
$x = 44$
Specific behaviours
✓ correct value

The difference in the balances of savings accounts A and B is least after y monthly payments have been made into each account.

(e) Determine the value of y and the difference between the balances of the accounts at this time. (2 marks)

Solution
$y = 25$ $a_y - b_y = 1690.09 - 1687.90 = \$2.19$
Specific behaviours
✓ correct value of y ✓ correct difference in balances

See next page

Question 12 **(7 marks)**

A software company is developing two smartphone apps called Sweety and Pics to estimate the percentage sugar content of common foodstuffs from photos taken by the phone's camera.

The following table shows estimates made using the Sweety app and the actual percentage sugar content for eight foodstuffs, where $r_{xy} = 0.799$.

Foodstuff	A	B	C	D	E	F	G	H
Estimate x	29	7	44	15	22	31	11	35
Actual y	57	4	58	29	43	31	32	45

(a) Using the estimate x as the explanatory variable, determine the equation of the least-squares line to model the linear relationship between x and y. (1 mark)

Solution
$\hat{y} = 1.1x + 10.7$
Specific behaviours
✓ correct equation

When the same eight foodstuffs were used to trial the Pics app, the equation of the least-squares line to predict the actual sugar content was $\hat{y} = 0.88x - 4.08$ and $r_{xy} = 0.91$.

(b) Does the Pics app tend to over or underestimate the actual sugar content? Justify your answer. (2 marks)

Solution
The app over-estimates the actual sugar content. The equation of the line indicates that actual values y will be smaller than estimated values x as the slope is less than 1 and the intercept is negative.
Specific behaviours
✓ states over-estimate ✓ justifies using coefficients of line

Sweety and Pics estimated the sugar content of a ninth foodstuff as 33% and 41% respectively.

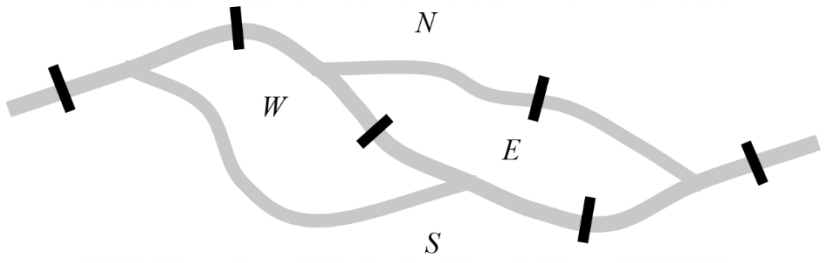
(c) Use the estimate from each app to predict the actual sugar content of the ninth foodstuff and explain which prediction you have more confidence in. (4 marks)

Solution
Sweety: $1.1(33) + 10.7 = 47\%$. Pics: $0.88(41) - 4.08 = 32\%$. More confident in the Pics prediction because by comparing the correlation coefficients, the estimates from Pics have a stronger association with the actual values than the Sweety app.
Specific behaviours
✓ one correct prediction ✓ second correct prediction ✓ chooses app with correlation coefficient closest to 1 ✓ indicates reasoning based on strength of correlation coefficients

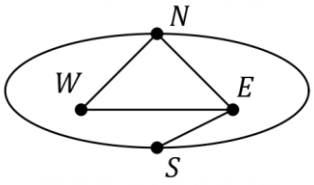
See next page

Question 13 **(9 marks)**

A river runs through a town, separating the northern region N from the southern region S. The river also splits, forming islands W and E, and six bridges cross the river at locations shown on the sketch map below.



(a) Use the information shown in the sketch map to draw a graph in the plane, in which vertices and edges represent the regions and the bridges respectively. (3 marks)

Solution

Specific behaviours
✓ four labelled vertices ✓ correct graph ✓ no crossed edges

(b) A tourist, staying in a hotel on island E, wants to take a morning run that starts from their hotel and crosses every bridge in the town once. They don't mind where their run ends. If possible, describe how they can do this. If not possible, explain why not. (2 marks)

Solution
It is possible as graph has an Euler trail that starts at E and ends at S. Example route: $E - N - S - E - W - N - S$.
Specific behaviours
✓ indicates jog possible ✓ lists suitable route, ending at S

(c) Planning is underway for a seventh bridge in the town. Explain how your answer to part (b) would change if the new bridge was a second river crossing between

(i) regions N and W. (2 marks)

Solution
Jog not possible - graph has four odd vertices and so no Euler trail or cycle exists.
Specific behaviours
✓ indicates jog not possible ✓ explains answer

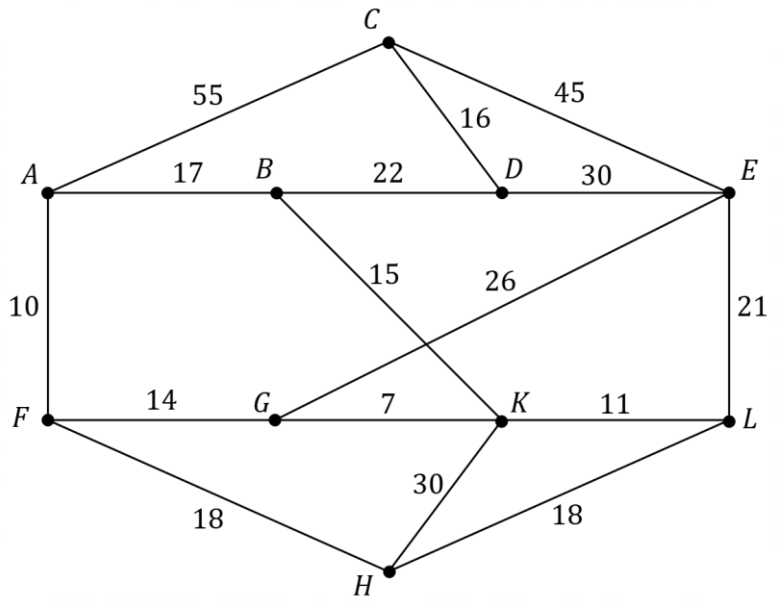
(ii) regions S and E. (2 marks)

Solution
Still possible - graph now has four even vertices and contains an Euler cycle. They can follow the same route as before, but once they reach S for the last time they can cross the new bridge and end up on island E.
Specific behaviours
✓ indicates jog still possible ✓ lists suitable route, ending at E, or explains difference from previous route

See next page

Question 14 **(9 marks)**

A company specialises in transporting sculptures, charging an amount equal to the cheapest path that exists between any pair of cities in the network shown below. Each edge weight represents the cost, in hundreds of dollars, to transport a sculpture along that edge.



(a) Determine how much the company will charge, and the associated path, to transport a sculpture between cities

(i) A and E. (3 marks)

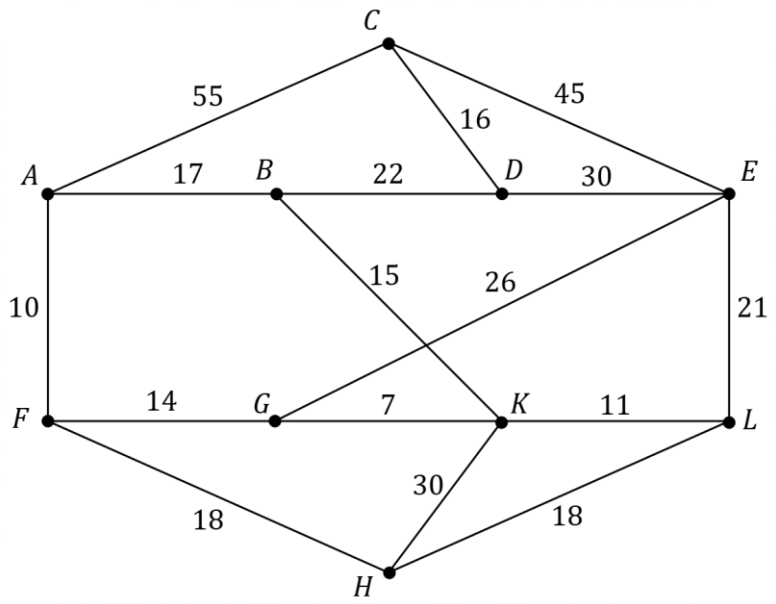
Solution
Example paths: $ABDE = 69, \quad AFGE = 50, \quad AFGKLE = 63$ Cheapest path is $AFGE$ and cost is $50 \times 100 = \$5000$.
Specific behaviours
✓ any correct path with cost ✓ correct cheapest path ✓ correct charge

(ii) C and H. (3 marks)

Solution
Cheapest path from C to $A = 55, B = 38, D = 16, E = 45, F = 65, G = 60, H = 82, K = 53, L = 64$. Example paths: $CAFH = 83, \quad CELH = 84, \quad CDBKH = 83, \quad CDBKLH = 82$ Cheapest path is $CDBKLH$ and cost is $82 \times 100 = \$8200$.
Specific behaviours
✓ any correct path with cost ✓ correct cheapest path ✓ correct charge

See next page

(b) Determine the increase in the amount charged to transport a sculpture between cities C and H if the company raised every cost shown on the network by $\$300$. A copy of the original graph has been provided below. (3 marks)



Solution
Cheapest path from C to $A = 58, B = 44, D = 19, E = 48, F = 71, G = 72, H = 92, K = 62, L = 72$. Example paths: $CAFH = 92, \quad CELH = 93, \quad CDBKH = 95$ Cheapest path now $CAFH$ and cost is $92 \times 100 = \$9200$, an increase of $\$1000$.
Specific behaviours
✓ indicates correct method (updates costs or modifies previous paths) ✓ indicates new cheapest path and cost ✓ states correct increase

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Question 15 **(12 marks)**

The number of years of experience t and the current salary s, in thousands of dollars, is shown in the table below for a sample of librarians who belong to a professional association.

t (years)	4	6	8	3	7	10	2	11	5
s ($000)	83	82	87	72	89	97	67	105	82

(a) Use your calculator to construct a scatterplot of the data with t as the explanatory variable and hence describe the nature of the relationship between the variables. (2 marks)

Solution
The scatterplot shows a strong, positive and linear relationship exists between the variables.
Specific behaviours
✓ describes nature using at least one of strength, direction or form ✓ describes nature using strength, direction and form

(b) Determine the equation of the least-squares line that can be used to predict s from t and state the correlation coefficient. (3 marks)

Solution
$\hat{s} = 3.63t + 62.31$ $r_{ts} = 0.960$
Specific behaviours
✓ uses supplied variables ✓ correct coefficients for line ✓ correct correlation coefficient

(c) Interpret, in the context of this question,

(i) the slope of the least-squares line. (2 marks)

Solution
For every extra year of experience, a librarian can expect their salary to increase by an average of $3630.
Specific behaviours
✓ refers to correct variables ✓ correctly states average salary increase

(ii) the intercept of the least-squares line. (1 mark)

Solution
The average salary of a librarian with no experience would be $62 310.
Specific behaviours
✓ correct interpretation

See next page

(d) What percentage of the variation in current salary of these librarians can be explained by the variation in number of years of experience? (1 mark)

Solution
$r^2 = 0.922$ and so 92% can be explained.
Specific behaviours
✓ correct percentage

(e) Another librarian who belonged to the same professional association has 15 years of experience.

(i) Use the least-squares line to predict the salary of this librarian. (1 mark)

Solution
$\hat{s} = 3.63(15) + 62.31 = 116.7$ Salary would be $\$116\,700$.
Specific behaviours
✓ correct prediction

(ii) Is the prediction in part (e)(i) reliable? Justify your response. (2 marks)

Solution
No – the prediction involves extrapolation beyond the range of data collected and so cannot be relied upon.
Specific behaviours
✓ responds with no ✓ justifies response by referring to extrapolation

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Question 16 **(9 marks)**

The volume of oil wasted, V_n litres, during the n^{th} trial of a new refining process is given by

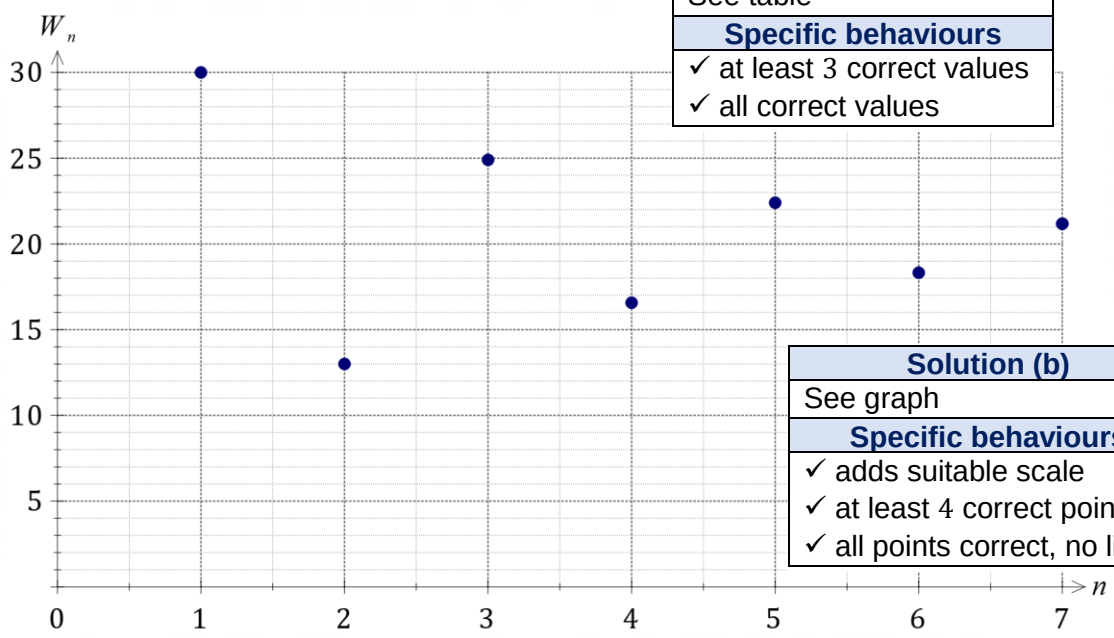
$$V_{n+1} = -0.7V_n + 34, \qquad V_1 = 30.$$

(a) Use the recurrence relation to complete the following table. (2 marks)

n	1	2	3	4	5	6	7
V_n (L)	30	**13**	**24.9**	**16.6**	**12.4**	**18.3**	21.2

(b) Add a suitable scale to the vertical axis and then display the volume wasted in each trial on the axes below. (3 marks)

Solution (a)
See table
Specific behaviours
✓ at least 3 correct values
✓ all correct values



Solution (b)
See graph
Specific behaviours
✓ adds suitable scale
✓ at least 4 correct points
✓ all points correct, no lines

(c) Will the volume wasted in the twelfth trial be more or less than in the tenth trial? Explain how you can answer this question without calculating W_{10} or W_{12}. (2 marks)

Solution
The volume will be more. The graph shows that the volume wasted always increases from any even numbered trial to the next even numbered trial.

Specific behaviours
✓ indicates weight will be more
✓ explanation using pattern from table or graph

(d) Describe how the volume wasted changes in the long term. (2 marks)

Solution
The volume wasted gradually converges to a long-term steady state value of 20 L.

Specific behaviours
✓ indicates convergence to steady state
✓ indicates correct long-term value

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Question 17 **(7 marks)**

The data in the following table shows x, the size of a country's iron ore stockpile in millions of tonnes and y, the price paid by that country to import iron ore in dollars per tonne.

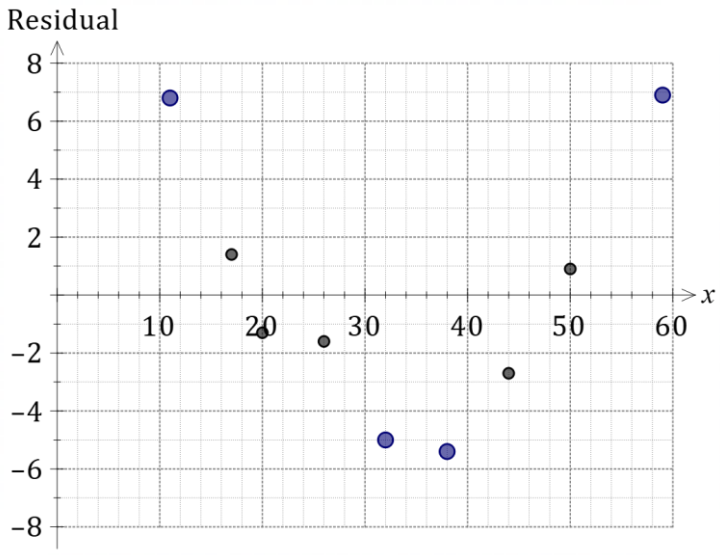
x (Mt)	11	17	20	26	32	38	44	50	59
y ($)	126	112	105	96	84	75	69	64	57

The data has a correlation coefficient of -0.982 and the equation of the least-squares line is $\hat{y} = 135.1 - 1.44x$.

(a) Show how to use the point $(20, 105)$ to calculate its residual of -1.3. (2 marks)

Solution
$\hat{y} = 135.1 - 1.44(20) = 106.3$ $\text{residual} = 105 - 106.3 = -1.3$
Specific behaviours
✓ shows calculation for $\hat{y}$ ✓ shows calculation for residual

(b) Complete the residual plot below. (3 marks)



Solution
Missing residuals: $126 - \hat{y}(11) = 6.8$ $84 - \hat{y}(32) = -5.0$ $75 - \hat{y}(38) = -5.4$ $57 - \hat{y}(59) = 6.9$ See graph
Specific behaviours
✓ indicates missing residuals ✓ correctly plots two residuals ✓ correctly plots all residuals

(c) Based on the residual plot, is it appropriate to fit a linear model to the data? Justify your answer. (2 marks)

Solution
No, it is not appropriate to fit a linear model to the data because a pattern is evident in the residual plot.
Specific behaviours
✓ states no ✓ justifies with reference to pattern in residual plot

See next page

Question 18 **(7 marks)**

Galatea has created graph G to represent a communications network within the company that she works for. The graph has $2x - 5$ faces and at least one vertex of degree x.

(a) Give two reasons why the value of x cannot be 2.5. (2 marks)

Solution
Any vertex degree x must be a whole number. The number of faces can't be $2(2.5) - 5 = 0$, but must be at least 1.
Specific behaviours
✓ states degree must be whole number ✓ indicates $f = 0$ and states must at least 1

Three more properties of G are that it is a connected planar graph, it has $2x + 3$ edges and it has $2x - 8$ vertices.

(b) Determine the value of x. (3 marks)

Solution
Connected planar graph means that $v + f - e = 2$. Hence $2x - 8 + 2x - 5 - (2x + 3) = 2$ $4x - 13 - 2x - 3 = 2$ $2x = 18$ $x = 9$
Specific behaviours
✓ indicates graph must obey Euler's formula ✓ correctly substitutes into Euler's formula ✓ solves for x

(c) State, with justification, whether G is Eulerian. (2 marks)

Solution
No – G has a vertex of degree 9, but to be Eulerian all its vertices must be even.
Specific behaviours
✓ states no, with justification ✓ justification

See next page

Question 19 **(8 marks)**

Farah suspected a small dam on her farm had a leak and began to monitor the depth of water every hour. The depth of water was initially 15.7 m and subsequent readings indicated it was decreasing by 0.6% every hour.

Let d be the depth of water in the dam t hours after monitoring began.

(a) Write an exponential equation in the form $d = ar^t$ to model the depth of water in the dam. (2 marks)

Solution
Decay rate: $r = 1 - 0.006 = 0.994$ $d = 15.7(0.994)^t$
Specific behaviours
✓ indicates correct value for a or r ✓ correct equation

(b) Determine the depth of water in the dam 3 days after monitoring began. (2 marks)

Solution
3 days is $3 \times 24 = 72$ hours. $d = 15.7(0.994)^{72} = 10.18$ m
Specific behaviours
✓ indicates correct value of t ✓ correct depth, to at least 1 dp

The leak was caused by a damaged valve. At the instant Farah fixed the valve, exactly 4 days after monitoring began, heavy rainfall in the region caused the depth of water in the dam to abruptly change from decreasing by 0.6% to increasing by 1.2% every hour.

(c) Determine the depth of water in the dam 6 days after monitoring began. (4 marks)

Solution
Depth when $t = 4 \times 24 = 96, d = 15.7(0.994)^{96} = 8.81$ m Growth rate: $r = 1 + 0.012 = 1.012$ New model for depth t hours after reaching this depth is $d = 8.81(1.012)^t$ Depth when $t = 48$ is $d = 8.84(1.012)^{48} = 15.62$ m.
Specific behaviours
✓ indicates depth at end of decreasing phase ✓ indicates growth ratio r ✓ indicates new equation to model increasing depth ✓ correct depth

End of questions

Supplementary page

Question number: _____

Supplementary page

Question number: _____